6.2

Change of variables formula

The substitution formula for functions of one variable is

$$\int_a^b f(x(u))\,\frac{dx}{du}\,du = \int_{x(a)}^{x(b)} f(x)\,dx$$

Here x is expressed in terms of u so we have a function x : R -> R which sends u to an expression x(u), and also a function f : R -> R defined where the variable is called x.

$$\underset{\text{variable } u}{\mathbb{R}} \xrightarrow{\;x\;} \underset{\text{variable } x}{\mathbb{R}} \xrightarrow{\;f\;} \mathbb{R}$$

On the left we are integrating the composite function f ∘ x

Because x probably does not change at the same speed as u , we have to include a factor dx / du

Example:

$$\int_0^2 2x e^{x^2}\,dx.\qquad \text{Put } u = x^2$$

$$du = 2x\,dx$$

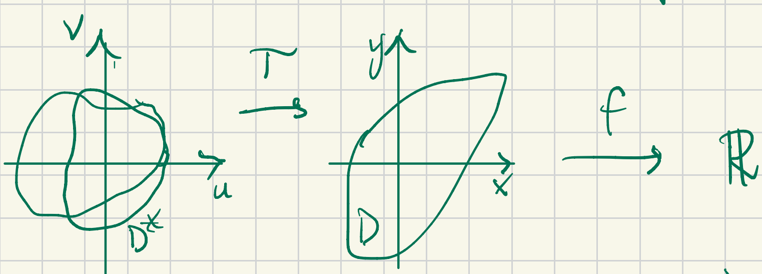
We get $\displaystyle\int_0^4 e^u\,du = e^4 - 1.$

In higher dimensions: the setup is a differentiable map $T : D^* \to D$.
We assume T is 1 - 1 and $D = T(D^*)$. Then

$$\iint_D f(x,y)\,dx\,dy$$

$$= \iint_{D^*} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

absolute value.



$$T(u,v) = (x(u,v), y(u,v))$$

Here $\dfrac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$

is the Jacobian = det of the Jacobian matrix.

Polar coordinates;

T : D* -> D is

T(r,t) = (x(r,t), y(r,t)) = (r cos t, r sin t).

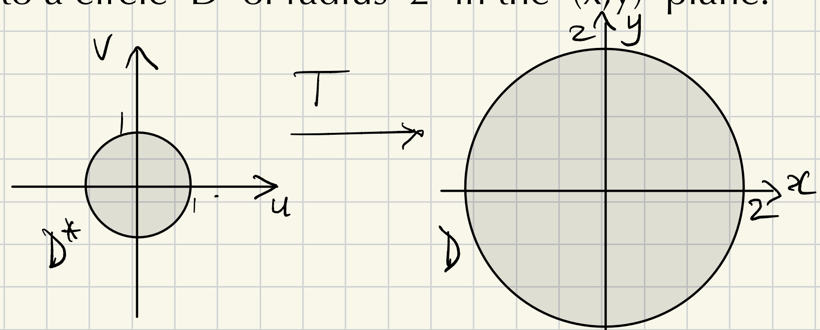
$$\frac{\partial(x,y)}{\partial(r,t)} = \det \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

$$\iint_D f(x,y) = \int_{D^*} f(r,\theta)\, r\, dr\, d\theta.$$

Pre-class Warm-up!!!

Let T : R^2 -> R^2 be the mapping
T(u,v) = (2u, 2v) and write (x,y) = T(u,v).
T maps the unit circle D* in the (u,v)-plane
to a circle D of radius 2 in the (x,y) plane.



Let f : R^2 -> R be a function.
What is the relationship between

$$A = \iint_D f \, dx\,dy = \iint_{D^*} f \circ T \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

$$B = \iint_{D^*} f \circ T \, du\,dv \qquad = 4B$$

Which of the following is true?

✓ a. A = 4B

b. A = 2B

c. A = B

d. A = B / 2

e. A = B / 4

Example: Let $f(x,y) = 1$ for all (x,y). Then $f(T(u,v)) = 1$ for all u, v.

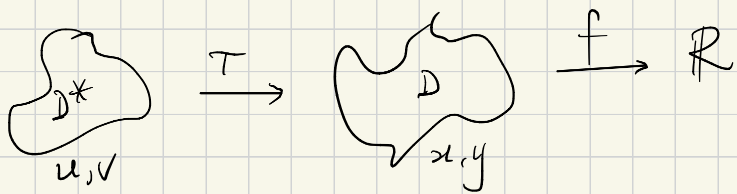
A = Area of D = 4B
B = Area of D*

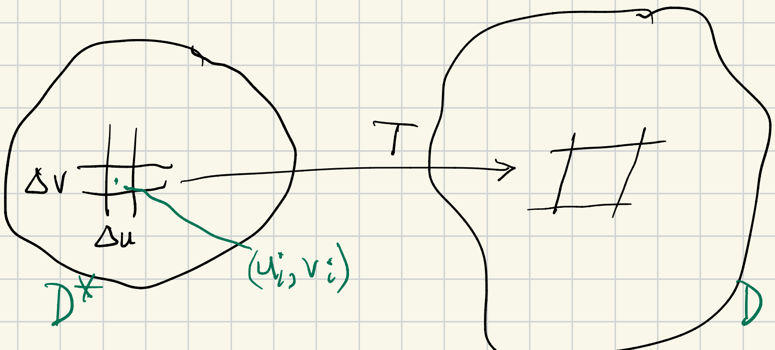
2nd approach $\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 4$

There will be no quiz tomorrow, or in the same week of Exam 3

Why does the change of variables formula work?

$$\iint_D f(x,y)\,dx\,dy = \int_{D^*} f(x(u,v),y(u,v))\,\frac{\partial(x,y)}{\partial(u,v)}\,du\,dv$$





The integral of $f \circ T$ over D^*

is approximated by the Riemann sum

$$\sum f(T(u_i,v_i))\,\Delta u\,\Delta v$$

$\iint_D f(x,y)\,dx\,dy$ is approximated by

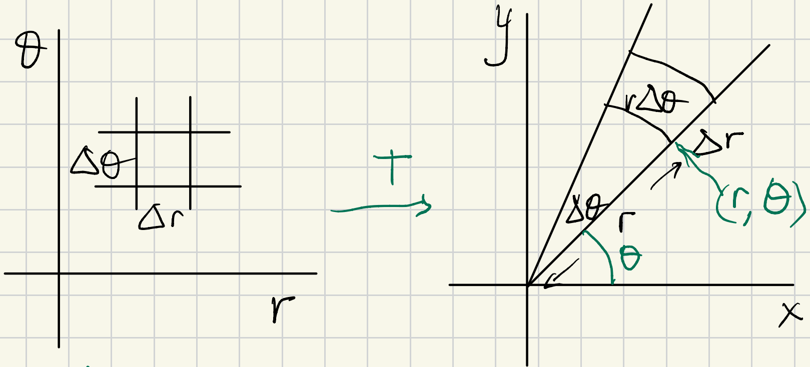
$$\sum_{\substack{\text{all parallegrams}\\ T(\text{little square})}} f(T(u_i,v_i))\,(\text{area of pgm})$$

$$= \sum_{\text{all pgms}} f(T(u_i,v_i))\,\Delta u\,\Delta v\,\frac{\partial(x,y)}{\partial(u,v)}$$

$$\approx \iint_{D^*} f\circ T\,\frac{\partial(x,y)}{\partial(u,v)}\,du\,dv$$

Areas multiply by $\frac{\partial(x,y)}{\partial(u,v)}$ on applying T.

Polar coordinates



$(x, y) = T(r, \theta)$.

$x = r\cos\theta \quad y = r\sin\theta$

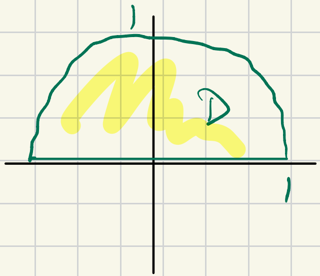
$$\iint f(x,y)\,dx\,dy = \iint f(r,\theta)\, r\,dr\,d\theta.$$

Under T the area $\Delta r \Delta\theta$ is sent to an area $r\Delta r\Delta\theta$

This explains the formula.

Example: Find $\iint_D x^2 y \, dx\,dy$

where D is the upper half of the unit disk



$x = r\cos\theta$

$y = r\sin\theta$

We use polar coords where $0 \le r \le 1$
$0 \le \theta \le \pi$ to get

$$\int_0^\pi \int_0^1 r^2\cos^2\theta \, r\sin\theta \, r\,dr\,d\theta$$

$$= \int_0^\pi \int_0^1 r^4 \cos^2\theta \sin\theta \, dr\, d\theta$$

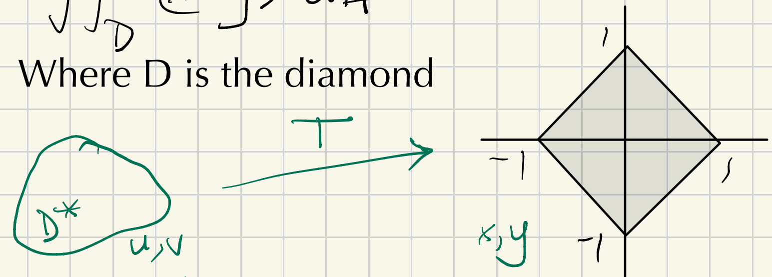
$$= \int_0^\pi \left[\frac{r^5}{5}\cos^2\theta\sin\theta\right]_0^1 d\theta = \int_0^\pi \frac{\cos^2\theta\sin\theta}{5}\, d\theta$$

$$= \left[-\frac{\cos^3\theta}{15}\right]_0^\pi =$$

Example. Use the transformation
T(u,v) = ((u+v)/2, (u-v)/2)) to find

$$\iint_D (x+y)^2 \, dA$$

Where D is the diamond



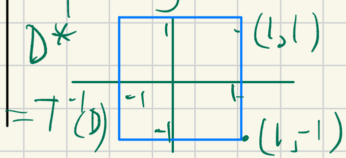
$x = (u+v)/2$ $\quad y = \frac{u-v}{2}$. $\quad \frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac12 & \frac12 \\ \frac12 & -\frac12 \end{bmatrix}$

$= -\frac{1}{2}$

To find D* we find T^{-1} by expressing u,v in terms of x,y: $\quad u = x+y$, $v = x-y$

if $(x,y) = (1,0)$, $(u,v) = (1,1)$,

if $(x,y) = (0,1)$, $(u,v) = (1,-1)$



$$\iint_{D^*} f\circ T \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \int_{-1}^{1}\int_{-1}^{1} u^2 \cdot \frac{1}{2}\, du\,dv$$

Change of variables for cylindrical coordinates

$x = r\cos\theta \quad y = r\sin\theta \quad z = z$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \left| \det \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \right|$$

$$= r(\cos^2\theta + \sin^2\theta) = r \quad \text{so}$$

$$dx\,dy\,dz = r\,dr\,d\theta\,dz$$

Find the volume bounded by
z= √(x^2 + y^2)
And   x^2 + y^2 + z^2 = 1

$\frac{\pi}{3}(2-\sqrt{2})$

Change of variables for spherical coordinates.

$x = \rho \sin\phi \cos\theta \quad y = \rho \sin\phi \sin\theta \quad z = \rho \cos\phi$

$$\iiint f \, dx\,dy\,dz = \iiint f \frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} d\rho\, d\theta\, d\phi$$

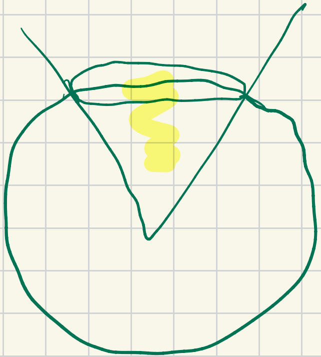
$$\frac{\partial(xyz)}{\partial(\rho\theta\phi)} = \left| \det \begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{bmatrix} \right|$$

$$= \left| \cos\phi\left(-\rho^2\sin\phi\cos\phi\sin^2\theta - \rho^2\sin\phi\cos\phi\cos^2\theta\right) + \cdots \right|$$

$$= \left| -\rho^2\sin\phi \right| = \rho^2\sin\phi$$

$$\iiint f \, dx\,dy\,dz = \iiint f \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

Find the volume bounded by
z= √(x^2 + y^2)
and   x^2 + y^2 + z^2 = 1



$$\iiint dV = \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^1 \rho^2\sin\phi \, d\rho\, d\theta\, d\phi$$

$$= \left[\frac{\rho^3}{3}\right]_0^1 \left[\theta\right]_0^{2\pi} \left[-\cos\phi\right]_0^{\pi/4}$$

$$= \frac{1}{3} \cdot 2\pi \cdot \left(1 - \frac{1}{\sqrt{2}}\right) = \frac{\pi}{3}\left(2 - \sqrt{2}\right)$$

Change of variables for spherical coordinates.

$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \left| \det \begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{bmatrix} \right|$$

$$= \left| \cos\phi\left(-\rho^2\sin\phi\cos\phi\sin^2\theta - \rho^2\sin\phi\cos\phi\cos^2\theta\right) - \rho\sin\phi\left(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta\right) \right|$$

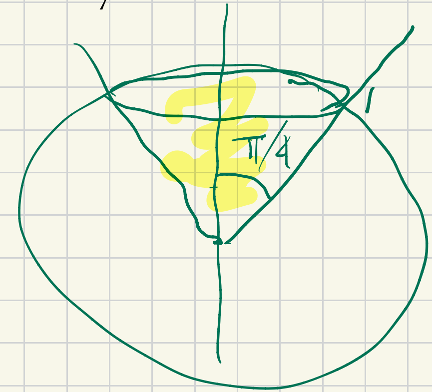
$$= \left| -\rho^2\sin\phi\left(\cos^2\phi\left(\sin^2\theta+\cos^2\theta\right)+\sin^2\phi\left(\sin^2\theta+\cos^2\theta\right)\right)\right|$$

$$= \left| -\rho^2\sin\phi \right| = \rho^2\sin\phi$$

Thus

$$dx\,dy\,dz = \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

Find the volume bounded by
z= √(x^2 + y^2)
and x^2 + y^2 + z^2 = 1



Volume $\iiint dx\,dy\,dz$

$$= \int_0^{\pi/4}\int_0^{2\pi}\int_0^{1} \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$$= \left[\frac{\rho^3}{3}\right]_0^1 \left[\theta\right]_0^{2\pi} \left[-\cos\phi\right]_0^{\pi/4} = \frac{1}{3}\cdot 2\pi \cdot \left(1-\frac{\sqrt{2}}{2}\right)$$